WEST BENGAL STATE UNIVERSITY

B.Sc. Honours 3rd Semester Examination, 2022-23

MTMACOR06T-MATHEMATICS (CC6)

GROUP THEORY I

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any ***five*** questions from the following: $2\times5=10$

(a) Let $\alpha=(1\ 2\ 4\ 6)$ and $\beta=(3\ 5\ 7)$ be two members of the symmetric group S_7. Find $\alpha\beta\alpha^{-1}$.

(b) Let $G=\langle a\rangle$ be a cyclic group of order 30. Find the order of the subgroup $\langle a^5\rangle$.

(c) Show that a group of order 119 can have atmost 112 elements of order 17.

(d) A binary operation $*$ on $\mathbb{Z}$ is defined by $m*n=2m+n$. Show that there is a left identity element but no right identity element.

(e) Find all the elements of order 4 in D_4, the dihedral group of order 4.

(f) Let H and K be the subgroups of a group G. Prove that the set $N_K(H)=\{x\in K: xH=Hx\}$ is a subgroup of G.

(g) Let $G=H\times K$ be the external direct product of two groups H and K. Prove that the set $S=\{(e,a): e$ is the identity of the group H and $a\in K\}$ is a normal subgroup of G.

(h) If $f=(i_1\, j_1)(i_2\, j_2)\cdots(i_k\, j_k)$ is a product of finite number of transpositions, find f^{-1}.

(i) If H and K are subgroups of a group G with $o(H)=18$ and $o(K)=35$. Find $o(H\cap K)$.

2. (a) Show that the set of all 2×2 real orthogonal matrices form a group with respect to matrix multiplication. 3

(b) Let $T=\{1,-1\}$ and $S=T\times T$. Let f and g be two bijections from S onto S defined by $f(x,y)=(x,-y)$ and $g(x,y)=(y,-x)$ for all $(x,y)\in S$. Prove that the set $G=\{f^i\circ g^i: i=0,1;\ j=0,1,2,3\}$ forms a group under the composition '$\circ$' of mappings, where $f^i=f\circ f\circ\cdots\circ f$ (i-times) and $f^0=$ the identity mapping on S. 5

3. (a) Suppose that a group G contains two elements a,b such that $o(a)=5$, $o(b)=2$ and $a^4b=ba$. Find the order of ab in G. 2

(b) In a group G, $(ab)^3=a^3b^3$ for all $a,b\in G$. Prove that the set $H=\{x^3: x\in G\}$ is a subgroup of G. 2

(c) Let G be a group and H a nonempty finite subset of G. Prove that H is a subgroup of G if and only if $ab\in H$, for all $a,b\in H$. 4

Turn Over

4. (a) Let $\sigma \in S_r (r \geq 2)$ and $\sigma = \sigma_1\sigma_2\sigma_3\cdots\sigma_k$ be a product of disjoint cycles in S_r. Suppose $o(\sigma_i) = n_i$, $i = 1, 2, \cdots, k$. Prove that $o(\sigma) = \text{lcm}(n_1, n_2, \cdots, n_k)$ in S_r. 4

(b) Let $\beta = (1\ 5\ 3\ 7)(9\ 6\ 8\ 4\ 2\ 10)$ in S_{10}. Find the smallest positive integer n such that $\beta^n = \beta^{-3}$. 2

(c) Let $\sigma = (1\ 3\ 7)(2\ 4\ 6\ 9)(5\ 8\ 10\ 11)$ and $\rho = (3\ 2\ 5\ 8)(4\ 7\ 10\ 1)(6\ 9\ 11)$ be two permutations in S_{11}. Find a permutation $\tau \in S_{11}$ such that $\rho = \tau\sigma\tau^{-1}$. 2

5. (a) Let H be a subgroup of a group G. For any $a \in G$, prove that the sets aH and H are equipotent. 2

(b) State and prove Lagrange's theorem for finite groups. 4

(c) Let p be a prime integer and a be an integer such that p does not divide a. Apply Lagrange's theorem to show that $a^{p-1} \equiv 1 (\text{mod } p)$. 2

6. (a) Prove that a finite group G of order n is cyclic if and only if it has an element of order n. 4

(b) Find all cyclic subgroup of the symmetric group S_3. 2

(c) Let G be a cyclic group of order 24 and $a \in G$. If $a^8 \neq e$ and $a^{12} \neq e$ then show that $G = \langle a \rangle$. 2

7. (a) Let $G = U_{16}$, the group of units modulo 16, $H = \{[1], [15]\}$ and $K = \{[1]; [9]\}$. Find G/H, G/K and HK. 3

(b) Show that a subgroup of index 2 is a normal subgroup. 2

(c) Let $H = \langle [8] \rangle$ in $\mathbb{Z}_{24}$. What is the order of $[14] + H$ in $\mathbb{Z}_{24}$? 3

8. (a) Define kernel of a group homomorphism. Show that the kernel is a normal subgroup of the domain. 1+3

(b) Show that the function $\phi : (\mathbb{R}, +) \to (S^1, \cdot)$ defined by $\phi(x) = e^{2\pi i x}$, $x \in \mathbb{R}$, is a group homomorphism, where S^1 is the multiplicative group of all complex numbers z with $|z| = 1$. Find the kernel of the homomorphism ϕ. 2+2

9. (a) Let G and G' be two finite groups and $f : G \to G'$ be a group homomorphism. Show for every $a \in G$, that $o(f(a))$ divides $o(a)$. 2

(b) Find the number of group homomorphisms from the cyclic group $\mathbb{Z}_{10}$ to the cyclic group $\mathbb{Z}_{21}$. 2

(c) Prove that any group of order 6 is either isomorphic to $\mathbb{Z}_6$ or to S_3. 4

10. (a) Let H and K be two subgroups of a group G. If K is normal in G, prove that 3

$$H/(H \cap K) \simeq (HK)/K.$$

(b) Show that $\mathbb{Z}_6$ is not a homomorphic image of $\mathbb{Z}_9$. 2

(c) Let G denote the Klein's 4-group. Find a subgroup H of the symmetric group S_4 such that G is isomorphic to H. 3

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